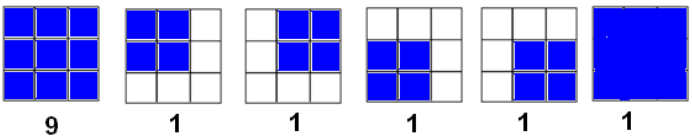
SOLUTIONS TO PRISM PROBLEMS

Senior Level 2011

1. (E)

As the diagram below indicates, there are 9 'one by one' squares, $1+1+1+1=4$ 'two by two' squares, and 1 'three by three' square. This gives a total of $9+4+1=14$ squares.



2. (C)

Let s be the distance from A to B. Also let t_1 be the time taken to get from A to B and let t_2 be the time taken to travel from B to A. Since speed is distance divided by time, we have $40 = s/t_1$ and $60 = s/t_2$. Hence $s = 40t_1 = 60t_2$, so $t_2 = \frac{2}{3}t_1$.

Also the total distance is the distance from A to B $+$ the distance from B to A $= 40t_1 + 60t_2$.

The average speed is the total distance divided by the total time, i.e. twice the distance from A to B divided by the total time, i.e. $\frac{2 \times 40t_1}{t_1 + t_2}$. Substituting $t_2 = \frac{2}{3}t_1$, we find that the average speed is $\frac{80t_1}{t_1 + \frac{2}{3}t_1} = \frac{80}{5/3} = 48$.

Note: A study of the argument above shows that in computing average speed we do not use the arithmetic average of the speeds from a to B and from B to A, but rather the harmonic mean, $\frac{2}{\frac{1}{\text{speed from A to B}} + \frac{1}{\text{speed from B to A}}}$, which can be written as $\frac{1}{\frac{\frac{1}{\text{speed from A to B}} + \frac{1}{\text{speed from B to A}}}{2}}$,

i.e. the reciprocal of the arithmetic mean of the reciprocals of the two speeds!

3. (E)

By writing each fraction as decimal, we see that $9/11 = 0.81818$ is the largest of the given fractions (each of the other fractions is at most 0.8).

4. (C)

Since there are 900 players and there is one winner, there must be 899 losers, each of whom plays exactly one game. Thus the total number of games played is 899.

Note: The above argument is of course much shorter than adding up the number of games played in the various rounds. If we do the latter, we get the same answer $450+225+112+56+28+14+7+4+2+1 = 899$.

5. (E)

For the red beads, we can choose $5, 4, 3, 2, 1$ or 0 red beads.

If we choose 5 red beads, we must then choose 0 blue and 0 white beads (one choice);

if we choose 4 red beads we can then choose either 1 blue and 0 white or 0 blue and 1 white (so 2 choices);

if we choose 3 red beads, we can then choose either 2 blue and 0 white, or 1 blue and 1 white or 0 blue and 2 white (so 3 choices);

if we choose 2 red beads, we can then choose either 3 blue and 0 white, or 2 blue and 1 white, or 1 blue and 2 white, or 0 blue and 3 white (so 4 choices);

if we choose 1 red bead, we can then choose either 4 blue and 0 white, or 3 blue and 1 white, or 2 blue and 2 white, or 1 blue and 3 white, or 0 blue and 4 white (so 5 choices);

and finally if we choose 0 red beads, we can choose 5 blue and 0 white, or 4 blue and 1 white, or 3 blue and 2 white, or 2 blue and 3 white, or 1 blue and 4 white, or 0 blue and 1 white (that is, 6 choices).

The total number of choices is then $1 + 2 + 3 + 4 + 5 + 6 = 21$.

Note: The above problem is an example of *combinations with repetition*, and there is a well-known formula for solving such problems. It uses the formula $\binom{n+r-1}{r}$ with $r = 5$ and $n - 3$. Before discussing this, we remind readers of the binomial coefficient.

The binomial coefficient is $\binom{m}{k} = \dfrac{m!}{k!(m-k)!}$ (some books denote this by ${}^{m}C_{k}$). This is an extremely important formula in an area of mathematics called *combinatorics*. The formula $\binom{m}{k}$ is referred to as *the number of combinations of m objects taken k at a time*. It has the following interpretations. It represents:

(1) the number of ways of placing m *distinguishable* objects into 2 cells such that the first cell contains k objects and the other cell contains the remaining $m - k$ objects;

(2) the number of ways of arranging m objects in a row when k of the objects are of one kind and the remaining $m - k$ are of a second kind;

(3) the number of groups, each of size r, that can be taken from m objects when order within any selected group is not important.

We now turn to the formula $\binom{n+r-1}{r}$. It has the following three representations:

(a) The number of ways of choosing r objects from n objects *when repetition is allowed;* order within any selected group is not important;

(b) The total number of ways of placing r (indistinguishable) objects into n cells;

(c) the number of n-tuples $(r_1, r_2, \ldots, r_n)$ of non-negative integers satisfying $\sum_{i=1}^{n} r_i = r$.

There is a standard "stars and bars" argument for deriving the formula $\binom{n+r-1}{r}$, but it can also be derived using more advance generating function methods. In the present example, we use interpretation (b) above and imagine placing $r = 5$ indistinguishable (identical in every way) beans into $n = 3$ boxes that are coloured red, blue and white. When a bean is placed in a box, the bean becomes the colour of that box. So for example, if we put 4 beans in the red box, 1 bean in the blue box and 0 beans in the white box, we would then say that we have chosen 5 red, 1 blue and 0 white beans. Applying interpretation (b) above, the number of ways of doing the distribution of beans is $\binom{n+r-1}{r} = \binom{3+5-1}{5} = \binom{7}{5} = \dfrac{7!}{5!2|} = \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1 \times 2 \times 1} = \dfrac{7 \times 6}{2 \times 1} = \dfrac{42}{2} = 21.$

6. (B)

An example in which (B) fails is to consider $2 \times 3 = 6.$ Here the product is even, but not both of 2 and 3 are even. It is easy to prove that each of (A), (C), (D) and (E) is always true.

7. (D)

One way is to note that $f(x^2+1) = x^4 + 2x^2 + 1 = (x^2+1)^2.$ Putting $y = x^2 + 1$ we have $f(y) = y^2$. Hence $f(x+2) = (x+2)^2$.

8. (E)

Note that $3 \times 33 = 99,$ so there are 33 two-digit numbers that have 3 as a factor, namely, $3, 6, 9, 12, \ldots, 99.$ Note also that $7 \times 14 = 98$ and there are 14 two-digit numbers that have 7 as a factor (these are $7, 14, 21, \ldots, 98$). The numbers that are divisible by *both* 3 and 7 are the four numbers $21, 42, 63,$ and 84. These four numbers are included in the above 33 numbers that have 3 as a factor and in the above 14 numbers that have 7 as a factor. Hence the number of 2-digit whole numbers that have either 3 or 7 as a factor is $33 + 14 - 4 = 43$ (note that we have subtracted 4 from $33 + 14$ because otherwise the numbers $21, 42, 63,$ and 84 would each have been enumerated twice).

9. (C)

Let A be Ann's current age and let B be Bob's current age. We want to find the difference $A - B$.

Suppose it was x years ago that Ann was three times as old as Bob. Then

$$A - x = 3(B - x) \qquad \textbf{(*)}$$

Also we are given that four years later, Ann was twice as old as Bob. Thus

$$A - x + 4 = 2(B - x + 4) \qquad \textbf{(**)}$$

Now (*) can be written as $A - x = 3B - 3x$ or

$$x = \frac{3}{2}B - \frac{1}{2}A \qquad \textbf{(***)}$$

Also, (**) is the same as $A - x + 4 = 2B - 2x + 8$, or

$$x = 2B - A + 4 \qquad \textbf{(****)}$$

From (***) and (****),we have

$\frac{3}{2}B - \frac{1}{2}A = 2B - A + 4.$

This is equivalent to $\frac{1}{2}B - \frac{1}{2}A = 4$, so $A - B = 8$

10. (A)

The triangle EDC is in fact an equilateral triangle. We were not expecting students to produce a proof of this within the time limits of the competition, but we hoped that they would use enlightened guesswork! A number of trigonometric and geometric proofs exist. Rather than exhibiting a proof here, we refer interested readers to e.g.

http://jwilson.coe.uga.edu/EMT668/EMAT6680.F99/Estes/15degrees/15degreeproblem

11. (E)

In the equation $x^2 - 2\sqrt{x^2} - 8 = 0,$ we set $y = \sqrt{x^2}$ (and by definition of the square root function, y is always non-negative). In terms of $y,$ the equation is $y^2 - 2y - 8 = 0.$ Factoring, we get $(y-4)(y+2) = 0,$ so $y = 4$ or $y = -2.$ But $y \geqslant 0$ so $y = 4$ only. Thus since $y = \sqrt{x^2}$, we have $\sqrt{x^2} = 4,$ and hence $x^2 = 16,$ so $x = 4$ or –4.

Note: The absolute value $|x|$ of any number x is the distance that number is from 0. For example, 7 and -7 are each a distance 7 away from 0, so $|7| = 7$ and $|-7| = 7.$

-7 0 7

In general, $|x| = x$ if $x \geqslant 0$ and $|x| = -x$ if $x < 0.$

Now note that asking to solve the equation $x^2 - 2\sqrt{x^2} - 8 = 0$ is the same as asking to solve $x^2 - 2|x| - 8 = 0,$ which is the same as solving $|x|^2 - 2|x| - 8 = 0.$ This gives $(|x|-4)(|x|+2) = 0,$ so $|x| = 4$ or $-2.$ But $|x|$ cannot be negative so we have $|x| = 4,$ and hence $x = 4$ or $-4,$ as above.

We note finally that if the question had asked to solve $x^2 - 2x - 8 = 0,$ we'd have $(x-4)(x+2) = 0$ so we'd get roots 4 and $-2.$ Solving the equation $x^2 - 2x - 8 = 0$ is quite different from solving $x^2 - 2\sqrt{x^2} - 8 = 0,$ because $2x$ may be negative or positive in general, whereas $2\sqrt{x^2}$ is always non-negative whether *or* not x is non-negative.

12. (A)

The arithmetic average of a set of n numbers is their sum divided by $n.$ Hence the sum of n numbers is n times their average. The 4 numbers whose average is 44 must then sum to $4 \times 44 = 156,$ and the 5 numbers whose average is 53 must sum to $5 \times 53 = 165.$ The average of all 9 numbers is then

$$\frac{\text{sum of the 9 numbers}}{9} = \frac{4 \times 44 + 5 \times 53}{9} = \frac{441}{9} = 49.$$

Note: It is instructive for students to note that the average, $\frac{4 \times 44 + 5 \times 53}{9},$ of all 9 numbers can be written in the form $\frac{4}{9} \times 44 + \frac{5}{9} \times 53.$ This is then a so-called *weighted average* of the numbers 44 and 53, with 'weights' 4/9 and 5/9 representing the proportion of all the numbers that made up the average 44 and the proportion of all the numbers that made up the average 53, respectively.

13. (A)

Recall that the Remainder Theorem says that when a polynomial is divided by $x - a$, the remainder is $f(a)$.

Thus with $f(x) = x^3 - 2x^2 + ax + b$, the given information implies that $f(1) = 1$ and $f(-2) = -8$.

The equation $f(1) = 1$ gives $(1)^3 - 2(1)^2 + a(1) + b = 1$, i.e.

$$a + b = 2 \qquad \textbf{(*)}$$

Also the equation $f(-2) = -8$ gives $(-2)^3 - 2(-2)^2 + a(-2) + b = -8$, so

$$-2a + b = 8 \qquad \textbf{(**)}$$

Solving the simultaneous equations (*) and (**) gives $a = -2$ and $b = 4$.

14. (D)

It is easy to see that an equilateral triangle with sides of length 1 has area $\frac{\sqrt{3}}{4}$. Also, a square with sides of length 1 has area 1, a circle with circumference of length 1 has area $\pi \times \left(\frac{1}{2\pi}\right)^2 = \frac{1}{4\pi}$, a circle with radius 1 has area π, and a square of diagonal length 1 has area $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2}$. Clearly the greatest of these five areas is π, so the circle of radius 1 has the largest area.

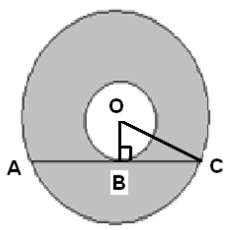
15. (E)

If r denotes the radius of the Earth, we are given that $2\pi r = 40,000,000$. The new rope has radius $r + h$ so its circumference is $2\pi(r + h) = 40,000,001$. Thus $2\pi r + 2\pi h = 40,000,001$. Hence since $2\pi r = 40,000,000$, we have $2\pi h = 1$, so $h = \frac{1}{2\pi}$, which is about 0.16 metres.

Note: The answer 0.16 is large in the sense that many people believe the rope will barely rise at all (after all, we added just 1 metre to a length of 40 million metres). The following observation emphasizes the beauty of the problem: "The answer is independent of the radius (and circumference) of the object about which the initial rope is wrapped (and incidentally this is an amazing fact!) Thus for example, the answer to the present problem is the same as the answer we would obtain if we added one metre to a rope that is circled around an orange, or the Sun!"

16. (A)

In the diagram shown below, the line OB is the perpendicular bisector of AC. The length of OB is the radius of the smaller circle, and the larger circle has radius |OC|.

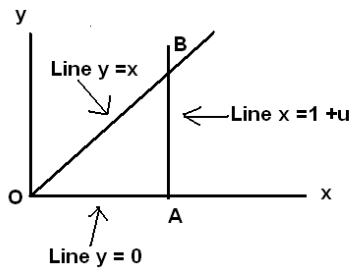


We are given that $|AC| = 10$, so $|BC| = 10/2 = 5$. By Pythagoras' Theorem, $|OC|^2 = |OB|^2 + |BC|^2$, so $|OC|^2 - |OB|^2 = 5^2 = 25$.

Multiplying across by π gives $\pi|OC|^2 - \pi|OB|^2 = 25\pi$, i.e. area of larger circle minus area of smaller circle $=$ area of ring $= 25\pi$.

17. (A)

In the diagram shown below, the point A has coordinates $(1+u, 0)$ and the point B has coordinates $(1+u, 1+u)$.



The area bounded by the lines $y = 0$, $y = x$ and $x = 1+u$ is the area of the triangle OAB which is $\frac{1}{2}|OA||AB|$. Thus $\frac{1}{2}(1+u)(1+u) = 8$, so $1+u = \sqrt{16}$, so $u = 3$ or -3. But $u > 0$ (given), and hence $u = 3$ only.

18. (D)

Here is one method of proceeding. Since we are told how long it takes 3 men and 6 boys to do the job, and want to calculate how long it would take a certain number of boys (without men) to do the job, the idea is to first work out what proportion of the job 3 men (without boys) would do in 2 hours.

Since two men can do the job in 4.5 hours, one man can do the job in $2 \times 4.5 = 9$ hours and 3 men could do it in $9/3 = 3$ hours. Hence 3 men could do one-third of the job in one hour, and thus

3 men can do two-thirds of the job in 2 hours **(*)**

We are given that

3 men and 6 boys can do the job in 2 hours **(**)**

We see from (*) and (**) that 6 boys do one-third of the job in 2 hours (the three men doing the other two-thirds of the job in that time).

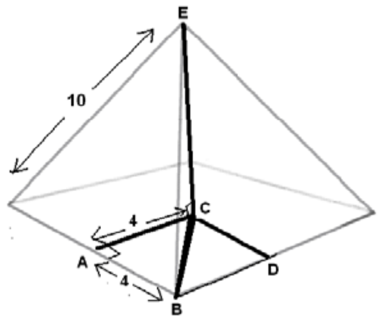
Since then 6 boys can do one-third of the job in 2 hours, we see that 6 boys do the entire job in $3 \times 2 = 6$ hours. Hence 2 boys (i.e. one-third of a team of 6 boys) can do the entire job in $3 \times 6 = 18$ hours.

19. (A)

The volume of a pyramid with rectangular base is $\frac{1}{3}\times$ (base area) $\times$ (perpendicular height)

(Note that this is one-third of the volume of a box that has the same base and height as the pyramid, just as the volume of a cone is one-third of the volume of a cylinder that has the same circular 'base' and 'height' as the cylinder).

In the diagram below, C denotes the centre of the base square, so the perpendicular height of the pyramid is $|CE|$. Also A and D are the centres of the two sides of the square base.



We first calculate $|BC|$ which is the length of the diagonal from corner B of the square base to the centre C of the base. Note that $|AB| = |AC| = 8/2 = 4$ and the angle BAC is a right angle. Then by Pythagoras' Theorem, $|BC| = \sqrt{|AB|^2 + |AC|^2} = \sqrt{4^2 + 4^2} = \sqrt{32}$.

Now we examine the triangle BCE which has a right angle at C. By Pythagoras' Theorem again, perpendicular height of pyramid $= |EC| = \sqrt{|EB|^2 - |BC|^2} = \sqrt{10^2 - 32} = \sqrt{100 - 32} = \sqrt{68}$.

Finally, the volume of the pyramid is $\frac{1}{3}\times$ (base area) $\times$ (perpendicular height) $= \frac{1}{3}\times 8^2 \times \sqrt{68} = \frac{648}{3}\sqrt{68}$.

20. (D)

We are given that $x + 75 = A^2$ and $x + 250 = B^2$ for integers A and B. Subtracting the first of these equations from the second gives $175 + A^2 . = B^2$. We try various squares $1^2, 2^2, 3^2, \ldots$ sequentially (and tediously!) until we find that $175 + (9)^2 = (16)^2$ and $175 + (15)^2 = (20)^2$. [Note that the positive integers $9, 15, 16$ and 20 can be replaced by $-9, 15, -16$ and -20, respectively, without affecting the calculations.]

We have $6 + 75 = (9)^2,\ 6 + 250 = (16)^2,\ 150 + 75 = (15)^2,\ 150 + 250 = (20)^2$.

The difference between the second-smallest and smallest values of x that satisfy the two given conditions is then $150 - 6 = 144$.
