

# SOLUTIONS TO PRISM PROBLEMS

## Junior Level 2011

### 1. (E)

$1 + 1 + 1 + 1 + 1 = 5$ , which is not 4.

### 2. (D)

The number of days in February is not 24, but rather 28, and 29 in leap years.

### 3. (E)

One way is to convert each fraction into a decimal (from which one will see that  $\frac{9}{11} = 0.81818\dots$  is the largest of the given fractions. Some students might get the lowest common multiple (LCM) of all of the denominators of all of the fractions. This however tedious because  $7 \times 4 \times 9 \times 5 \times 11 = 13,860$  is the smallest number into which all of the denominators of the fractions will go.

### 4. (B)

Any selection of three beads will contain either all red beads, all blue beads, 2 red beads and 1 blue bead, or else 2 blue beads and 1 red bead. There are thus

4 ways in all.

*Note on distinguishable beads:* If the beads were distinguishable (e.g. if each bead had a unique number written on it), we'd need to know the total number of beads of each colour, and we'd then perform the enumeration by partitioning as above and using the product of binomial coefficients to evaluate each term.

Aside: The binomial coefficient  $\binom{n}{r}$ , sometimes denoted  $nCr$ , is given by the formula  $\binom{n}{r} = \frac{n!}{r!(n-r)!}$  or equivalently as  $\frac{n(n-1)\dots(n-r+1)}{1 \times 2 \times \dots \times r}$ . It has a number of representations, including being the number of ways of choosing a committee of  $r$  people from  $n$  people when order of individuals within each selected committee is not important. Of course,  $m! = m(m-1)(m-2)\dots(3)(2)(1)$  represents the number of ways of arranging  $m$  distinct objects in a row.

Now suppose the beads are distinguishable and that there are, for example, 5 red and 6 blue beads in all. Then the number of ways of getting all red beads would be the number of ways of choosing 3 red beads from 5 red beads and no

blue bead from the 6 blue beads. This would be

$$\binom{5}{3} \binom{6}{0} = \binom{5}{3} \times 1 = \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 10. \text{ One would}$$

similarly evaluate the number of ways of getting all blue beads to be 20, 2 red beads and 1 blue bead to equal 60, and 2 blue beads and 1 red bead as 75.

Thus if there were 5 red and 6 blue beads in all and if all beads were distinguishable, the number of different selections of three beads would then be  $10 + 20 + 60 + 75 = 165$ .

### 5. (B)

Let  $A$  denote Anne's current age and let  $B$  be Bob's current age. Since Anne is five years older than Bob, we have  $A = B + 5$ . Also, since two years ago, Ann's age was  $A - 2$  and Bob's age was  $B - 2$ , we have  $A - 2 = 2(B - 2)$ .

One way of proceeding is now to substituting  $A = B + 5$  into  $A - 2 = 2(B - 2)$ , thus eliminating one of the unknowns  $A$ . This gives  $B + 5 - 2 = 2(B - 2)$ . We now solve this equation for  $B$ . We have  $B + 3 = 2B - 4$ , so  $3 + 4 = 2B - B$ , so  $B = 7$ . From  $A = B + 5$ , we then get  $B = 7 + 5 = 12$ .

### 6. (B)

Let  $x$  be the number of questions on the English test. Since Pat got 40% in the maths test, he must have got 40% of 10 questions correct, i.e. he got 4 questions right on the Maths test. Hence he also got 4 questions correct on the English test. . Since he scored 80% on the English test, we have that 60% of the number of English test questions equals 4, so the number of English test questions is 5.

### 7. (B)

Let  $l$  denote the length of a longer side and  $s$  is the length of a shorter side. We are given that the perimeter is 36, so  $l + l + s + s = 36$ , i.e.  $2l + 2s = 36$  or  $l + s = 18$ . We are also given that the sum of the lengths of the two longer sides is equal to eight times the sum of the lengths of the two shorter sides, i.e. that  $2l = 8 \times 2s$ , i.e.  $l = 8s$ . Substituting  $l = 8s$  into  $l + s = 18$ , we get  $8s + s = 18$ , so  $s = 2$ .

### 8. (C)

If  $c$  denotes the length of the hypotenuse of a right-angled triangle and  $a$  and  $b$  denote the lengths of the other two sides, we have by Pythagoras' Theorem that  $c^2 = a^2 + b^2$ . It is easy to see that of the five given choices, only choice (C) 2,3,4

fails this equation. In fact  $4^2 = 16$  while  $2^2 + 3^2 = 4 + 9 = 13$ .

**9. (B)**

The two-digit multiples of 3 and 7 are  $3 \times 7 = 21$ ,  $2 \times 3 \times 7 = 42$ ,  $3 \times 3 \times 7 = 63$  and  $4 \times 3 \times 7 = 84$ , so the answer is 4. Note that the next multiple is  $5 \times 3 \times 7 = 102$  but this is not a two-digit number.

**10. (C)**

If  $a$  is the length=width=height of the cube, we have  $a^3 = 24$ . If we halve each of the three dimensions, the new volume will be  $\left(\frac{1}{2}a\right)^3 = \frac{a^3}{8} = \frac{24}{8} = 3$ .

**11. (E)**

We see that each of the three given points  $(0,10)$ ,  $(5,0)$  and  $(10,-10)$  fall on the line  $y = -2x + 10$ , because  $0 = -2(10) + 10$ ,  $0 = -2(5) + 10$  and  $-10 = -2(10) + 10$ . However, one can also check that not all of the the three points falls on any of the other four given lines. For example, the point  $(0, 10)$  does not fall on the line  $y = x - 20$  because  $10 \neq 0 - 20$ .

**12. (D)**

Let  $A$  and  $B$  denotes the amounts of money possessed by Ann and Breda, respectively. We are given that  $A + B = 21$  and  $A - 2 = 2(B + 2)$ , i.e.  $A - 2B = 6$ . Solving these two simultaneous equations gives  $A = 16$  and  $B = 5$ .

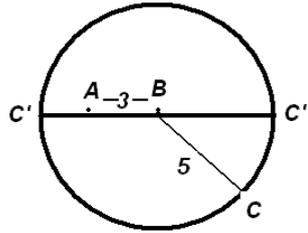
**13. (E)**

There are an infinite number of circles. Start with any point  $O$  on  $L$  and draw a circle of radius length  $\|UO\|$ , the distance between  $U$  and  $O$ . Note that this circle will go through the point  $V$  because  $V$  and  $U$  are equidistant from the line  $L$  and hence from  $O$ .

*Note:* Students who think the answer to this question is 1 presumably believe that the only possible circle is one whose centre is at the point of intersection of the line  $L$  and the line joining  $U$  and  $V$ .

**14. (E)**

Since Bart and Charlie live 5 miles apart, Charlie's home can be located anywhere on the circle of radius 5 shown below.

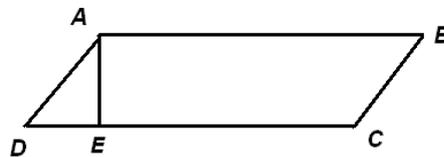


The points on the circle all lie somewhere between 2 and 8 kms from point A. For the nearest and farthest distances of C from A, notice that if Charlie lives at point C', he would be only 2 kms from Andy, while if he lives at point C'', he would be 8 kms from Andy.

**15. (E)**

The equation  $\frac{4^{-m}}{27} = \frac{3^{-n}}{16}$  is the same as  $\frac{4^{-m}}{3^3} = \frac{3^{-n}}{4^2}$  or  $4^{-m} \times 4^2 = 3^3 \times 3^{-n}$ , so  $4^{2-m} = 3^{3-n}$ . Since the left side is a power of an even number and the right side is a power of an odd number, the only way the identity can hold true is if each of the two exponents  $2 - m$  and  $3 - n$  is 0. Thus  $m = 2$  and  $n = 3$ .

**16. (E)** We drop a perpendicular  $AE$  from  $A$  to  $DC$ .



Then  $\sin(\angle ADE) = \frac{\|AE\|}{\|AD\|}$  and we now first calculate each of the lengths  $\|AE\|$  and  $\|AD\|$ .

Since  $\|AB\| = 3\|AD\|$  and the perimeter of ABCD is 80m, we have  $2\|AD\| + 2 \times 3\|AD\| = 80$ , so  $\|AD\| = 10$  and  $\|AB\| = 3 \times 10 = 30$ .

Since also the area of ABCD is 200, we have  $\|AE\| \times \|AB\| = 240$ , so  $\|AE\| = \frac{240}{30} = 8$ .

Hence finally  $\sin(ADE) = \frac{\|AE\|}{\|AD\|} = \frac{8}{10} = \frac{4}{5}$

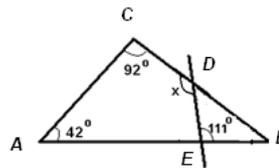
**17. (D)** For every metre run by Ann, Betty will run  $\frac{180}{200} = 0.9$  of a metre. For every metre run by Betty, Cathy runs  $\frac{150}{200} = 0.75$  of a metre. Hence for every metre run by Ann, Cathy will run  $0.9 \times 0.75 = 0.675$  of a metre. Hence when

Ann has reached the finishing line (i.e. covered 200m), Cathy will have covered  $0.675 \times 200 = 135$  metres. Thus Ann beat Cathy by  $200 - 135 = 65$  metres.

**18. (A)**

Since the sum of the angles in any triangle is  $180^\circ$ , we have from the triangle ABC that the angle ABC must be  $180 - 92 - 42 = 46$ . Then in the triangle EBD, we have  $111 + 46 + \text{angle } EDB = 180$ , we have angle  $EDB = 23$ .

Hence  $x = 180 - 23 = 157$ .



**19. (E)**

Let  $x$  be the speed of the cyclist in still air and let  $w$  be the speed of the wind. We want to evaluate the ratio  $\frac{x}{w}$ .

The actual speed of the cyclist in the direction of the wind is then  $x + w$  and the actual speed of the cyclist going against the wind is  $x - w$ . Recall that average

$$\text{speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

so

$$\text{distance travelled} = (\text{average speed})(\text{time taken}) \quad (*)$$

Since the distance the cyclist travels from A to B is the same as from B to A, we have from (\*)

$$(\text{speed of cyclist from A to B})(\text{time taken to travel from A to B}) =$$

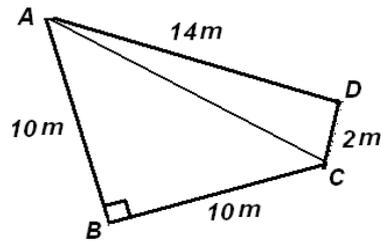
$$(\text{speed of cyclist from B to A})(\text{time taken to travel from B to A}).$$

$$\text{Thus, } (x + w)30 = (x - w)40.$$

$$\text{Hence } 10x = 70w, \text{ and so } \frac{x}{w} = 7.$$

**20. (A)**

One way of proceeding is to first construct the line AC shown below.



Since the triangle  $ABC$  has a right-angle at  $B$ , we have by Pythagoras' Theorem,  $\|AC\|^2 = (10)^2 + (10)^2 = 200$ . Now in the triangle  $ACD$ , note that the sum of squares of the lengths of the sides  $CD$  and  $AD$  is  $\|CD\|^2 + \|AD\|^2 = (2)^2 + (14)^2 = 4 + 196 = 200$ . But as just shown, 200 is the squared length of  $AC$ . Thus the triangle  $ACD$  is also a right-angle triangle. We recall that the area of a right-angle triangle is one-half the base times the perpendicular height.

The area of the quadrilateral is then the sum of the areas of the triangles  $ABC$  and  $ACD$ , and this is

$$\frac{1}{2} \times 10 \times 10 + \frac{1}{2} \times 2 \times 14 = 64 \text{ m}^2.$$


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